Cubic Transmuted Inverse Rayleigh Distribution for Modeling Breast Cancer Data

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Abstract
This article presents a new three-parameter statistical distribution called Cubic Transmuted Inverse Rayleigh distribution derived using cubic transmutation map suggested by Aslam et al. (2018). Various statistical properties of this distribution were investigated which includes: Hazard function, moments, moment generating function, order statistics, Renyi entropy were obtained. The maximum likelihood estimator of the unknown parameters of the distribution was derived. Application to real data set shows it tractability over its sub-models in analyzing life data.

Keywords: *Moment, Likelihood estimation, Cubic Transmutation Map, Moment generating function*

1.0 Introduction

Trayer (1964) developed a flecible lifetime distribution known as inverse Rayleigh (IR) distribution. Specially, it provides a unique statistical model when dealing with unimodal highly right-skewed data. As mathematical basis, the corresponding cumulative distribution function (cdf) and probability density function (pdf) are, respectively, given by

$$F(x;\theta) = e^{-(\theta x)^{-2}}; \qquad x > 0, \theta > 0 \tag{1}$$

and

$$f(x;\theta) = \frac{2}{\theta^2 x^3} e^{-(\theta x)^{-2}}; \qquad x > 0, \theta > 0 \tag{2}$$

Where θ is a scale parameter. As notable features, the IR distribution has tractable and simple probability functions, is unimodal and right-skewed, and possesses a hazard rate function which shows a singular curvature: it increases at a certain value, then decreases until attain a kind of stabilization. The early researcher who studied the IR distribution are Voda (1972) which studied some properties of the maximum likelihood estimator of, and Gharraph (1993) which provides closed-form expressions for the mean, harmonic mean, geometric mean, mode and median of the IR distribution. In the recent years, many extensions of the IR distribution were developed, using different statistical techniques, majorly on the basis of general families of distributions. Among them, there are the beta Inverse Rayleigh distribution by Leao et al. (2013), Transmuted Inverse Rayleigh distribution by Ahmad and Ahmad (2014), modified Inverse Rayleigh distribution by Khan (2014), transmuted modified Inverse Rayleigh distribution by Khan and King (2015), transmuted exponentiated Inverse Rayleigh distribution by Haq (2015), Kumaraswamy exponentiated Inverse Rayleigh distribution by Haq, (2016), weighted Inverse Rayleigh distribution by Fatima and Ahmad (2017), odd Fréchet Inverse Rayleigh distribution by Elgarhy and Alrajhi (2019), type II Topp-Leone Inverse Rayleigh distribution by Mohammed and Yahia

(2019), type II Topp-Leone generalized Inverse Rayleigh distribution by Yahia and Mohammed (2019) and exponentiated Inverse Rayleigh distribution by Rao and Mbwambo (2019). This study focuses on developing a more flexible Inverse Rayleigh distribution called Cubic Transmuted Inverse Rayleigh (CTIR) distribution which is applicable in modeling lifetime data.

2.0 Generalised Transmuted Family of Distributions

A general transmuted family of distributions that can be used to generate new families. Let X be a random variable with cdf $F(x)$, then a general transmuted family; called k-transmuted family; is defined by

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^{k} \lambda_i [G(x)]^i \quad (3)$$

With $\lambda_i \in [-1,1]$ for $i = 1,2,. . ., p$ and $-k \leq \sum_{i=0}^{l} \lambda_i \leq 1$. The general transmuted family reduces to the base distribution for $\lambda_i = 0$, for $i = 1,2,3,. . ., l$. The density function corresponding to (3) is

$$f(x) = g(x)\left\{1 - \sum_{i=1}^{l} \lambda_i [G(x)]^i + [1 - G(x)] \sum_{i=1}^{l} \lambda_i G^i(x)\right\} \quad (4)$$

The density function of the quadratic transmuted quadratic family of distributions which was obtained by letting $l = 1$ as defined by Shaw et al. (2007), is given as

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x) \quad (5)$$

Where $\lambda \in (1, -1)$ is the transmutation parameter. The quadratic transmuted family of distribution given in (4) has a wider area of applications to any baseline $G(x)$. The quadratic transmuted distribution does not provide a reasonable fit if the data is bi-modal in nature.

To solve the problem of bi-modality encountered in real data, the cubic transmuted family of distributions is obtained by setting $k = 2$ in (2), which was showed in the work of Aslam Mohammad et al. (2017), and is given as

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) + (1 - \lambda_2)G^3(x) \quad (6)$$

Where $\lambda_1 \in [-1,1]$, $\lambda_2 \in [-1,1]$ are the transmutation parameters and consequently obey the condition

$-2 \leq \lambda_1 + \lambda_2 \leq 1$.

The pdf corresponding to the equation (5) is defined as

$$f(x) = g(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1 - \lambda_2)G^2(x)] \quad (7)$$

This method has been used by several authors to generalize a distribution for better adaptability to data. For examples Ogunde et al. (2020), Rahman et al. (2018), Bugra Sracoglu and Caner Tams (2018).

2.1 Cubic Transmuted Inverse Rayleigh distribution

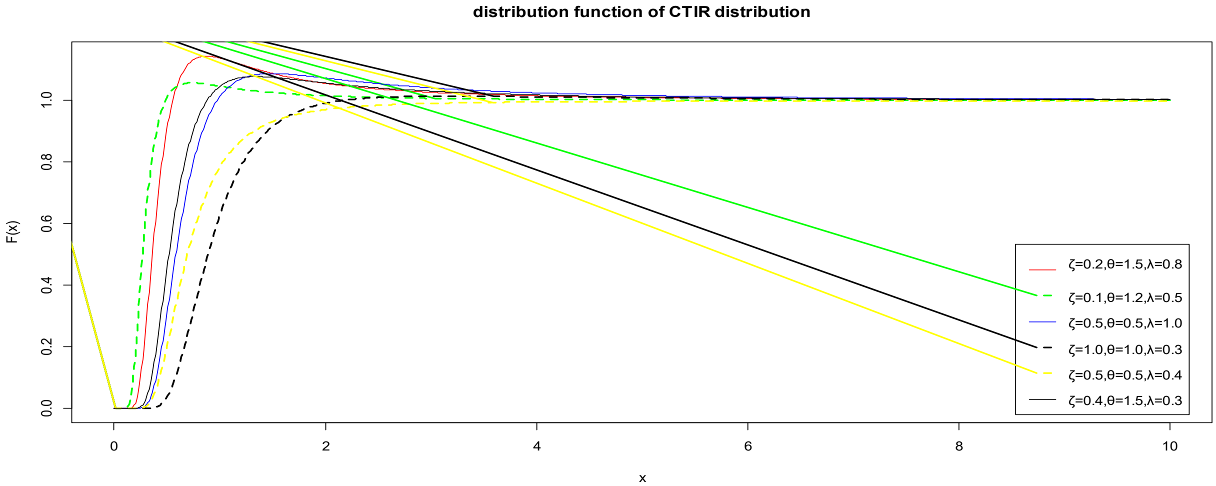
A random variable X is said to follow the $CTIR$ distribution if its cdf is given by

$$F(x;\gamma) = \left[\lambda_1 e^{-(\theta x)^{-2}} + (\lambda_2 - \lambda_1)\left(e^{-(\theta x)^{-2}}\right)^2 + (1 - \lambda_2)\left(e^{-(\theta x)^{-2}}\right)^3\right] \quad (8)$$

And the associated pdf is given by

$$f(x;\gamma) = \frac{2}{\theta^2 x^3} e^{-(\theta x)^{-2}} \left[\lambda_1 + 2(\lambda_2 - \lambda_1)e^{-(\theta x)^{-2}} + 3(1 - \lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right] \quad (9)$$

Figure 2.0 drawn below shows the shape of the density function of the $CTIR$ distribution from arbitrary values of the parameters. The graph shows that the pdf of $CTIR$ is unimodal and right skewed with different degrees of kurtosis.



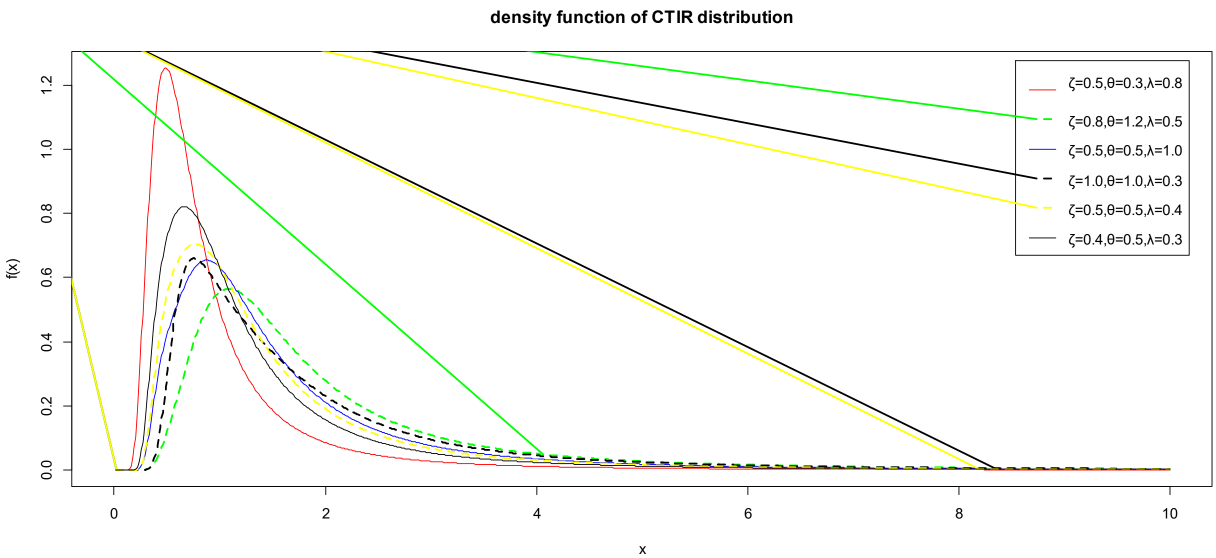


Figure 2: The Graph of the density function of $CTIR$ distribution

- ✓ Figure 2.0 drawn above illustrates the properties of $CTIR$ distribution being able to model both the unimodal and bimodal data.

The survival and the hazard rate function of the $CTIR$ distribution are respectively given by

$$S(x;\gamma) = 1 - \left[\lambda_1\left(e^{-(\theta x)^{-2}}\right) + (\lambda_2 - \lambda_1)\left(e^{-(\theta x)^{-2}}\right)^2 + (1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^3\right] \quad (10)$$

and

$$h(x;\gamma) = \frac{\frac{2}{\theta^2 x^3} e^{-(\theta x)^{-2}}\left[\lambda_1 + 2(\lambda_2 - \lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right]}{1 - \left[\lambda_1\left(e^{-(\theta x)^{-2}}\right) + (\lambda_2 - \lambda_1)\left(e^{-(\theta x)^{-2}}\right)^2 + (1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^3\right]} \quad (11)$$

Figure 3 respectively are the graph of the survival function and hazard function of $CTIR$ distribution for various values of the parameters.

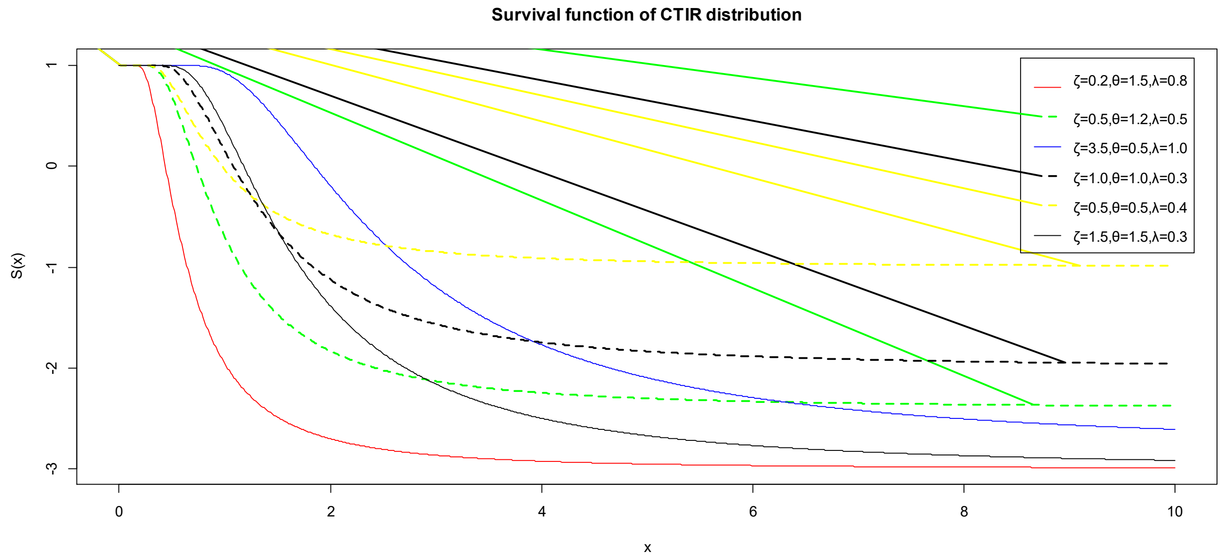


Figure 3: The Graph of the survival function of $CTIR$ distribution

3.0 Moments

Moment of a distribution plays a very important role in statistical analysis. They are used for estimating features and characteristics of a distribution such as skewness, kurtosis, measures of central tendency and measures of dispersion.

LEMMA 3.1. $If\ X \sim CTE(\gamma)$, where $\gamma = \{\theta, \lambda_1, \lambda_2\}$, then the r^{th} non-central moment of X is given by

$$E(X^r) = \mu_r' =$$

Proof.: By definition, the $\boldsymbol{r^{th}}$ non-central moment is given by

$$\mu_r' = \int_0^\infty x^r f(x;\gamma)dx \tag{12}$$

Putting (8) in (12), we have

$$\mu_r' = \frac{2}{\theta^2}\int_{-\infty}^{\infty} x^{r-3}\, e^{-(\theta x)^{-2}}\left[\lambda_1 + 2(\lambda_2-\lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right] dx \tag{13}$$

From equation (13), taking

$$W_1 = \lambda_1\frac{2}{\theta^2}\int_{-\infty}^{\infty} x^{r-3}\, e^{-(\theta x)^{-2}}dx = \lambda_1\theta^{-r}\Gamma\left(\frac{r}{2}+1\right), \tag{14}$$

$$W_2 = \frac{4(\lambda_2-\lambda_1)}{\theta^2}\int_{-\infty}^{\infty} x^{r-3}\left(e^{-(\theta x)^{-2}}\right)^2 dx = \theta^{-r}(\lambda_2-\lambda_1)\left(2-2^{-r/2}\right)\Gamma\left(\frac{r}{2}+1\right) \tag{15}$$

$$W_3 = \frac{6(1-\lambda_2)}{\theta^2}\int_{-\infty}^{\infty} x^{r-3}\left(e^{-(\theta x)^{-2}}\right)^3 dx = 3\theta^{-r}\lambda_2\sum_{i=1}^{\infty}\binom{2}{i}(-1)^i(i+1)^{-\left(\frac{r}{2}+1\right)}\Gamma\left(\frac{r}{2}+1\right) \tag{16}$$

Combining equation (14), (15) and (17), we obtain the r^{th} moment of CTF distribution given as:

$$\mu_r' = \theta^{-r}\Gamma\left(\frac{r}{2}+1\right)\left\{\lambda_1 + (\lambda_2-\lambda_1)\left(2-2^{-r/2}\right) + 3\lambda_2\sum_{i=1}^{\infty}\binom{2}{i}(-1)^i(i+1)^{-\left(\frac{r}{2}+1\right)}\right\} \quad (17)$$

For $r = 1, 2, \ldots, \Gamma(.)$ is the gamma function.

3.1 Moment generating function

Moment Generating Functions (MGF): These are special functions that are used to obtain the moments and its functions such as: mean and variance of a random variable in a simpler way. LEMMA 3.2. If $X \sim CTIR\ (\gamma)$, where $\gamma = \{\zeta, \gamma, \lambda_2, \lambda_1\}$, then the $MGIR$ of X is given by
Proof: By definition, the MGF is given by

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x;\gamma)dx \quad (18)$$

Using the series expansion e^{tx} gives

$$M_X(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty} x^k f(x)dx = \sum_{k=0}^{\infty}\frac{t^k\mu_k'}{k!} \quad (19)$$

Putting equation (17) in (19), we have

$$M_X(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\theta^{-r}\Gamma\left(\frac{r}{2}+1\right)\left\{\lambda_1 + (\lambda_2-\lambda_1)\left(2-2^{-r/2}\right) + 3\lambda_2\sum_{i=1}^{\infty}\binom{2}{i}(-1)^i(i+1)^{-\left(\frac{r}{2}+1\right)}\right\} \quad (20)$$

3.2 Entropy

Entropies is a measure of randomness of a system and has been used extensively in information theory. Two popular entropy measures are Renyi entropy (1961) and Shannon entropy (1948). A large value of the entropy indicates a greater uncertainty in the data. The Shannon entropy is a special case of the Renyi entropy when $\rho \to 1$ and is given by $E[-\log(f\ (x;\gamma))]$.

LEMMA 3.3: If the random variable X has a CTF distribution, then the Renyi entropy of X is given by

$$E_R(\rho) = \frac{1}{1-\rho}\log\left[\frac{2}{\theta^2}\int_0^{\infty}\left\{x^{-3}e^{-(\theta x)^{-2}}\left[\lambda_1 + 2(\lambda_2-\lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right]\right\}^{\rho}dx\right]$$

Proof: By definition

$$E_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^{\infty} f^z(x)dx\right\}, z > 0, z \neq 0. \quad (21)$$

Putting equation (8) in (20), we have

$$E_R(z) = \frac{1}{1-\rho}\log\left[\frac{2}{\theta^2}\int_0^{\infty}\left\{x^{-3}e^{-(\theta x)^{-2}}\left[\lambda_1 + 2(\lambda_2-\lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right]\right\}^{\rho}dx\right]$$

The equation above can be written as

$$E_R(\rho) = \frac{1}{1-\rho} log\left\{\int_0^\infty {I_1}^\rho dx * \int_0^\infty {I_2}^\rho dx\right\} \tag{22}$$

Where

$$\int_0^\infty {I_1}^\rho dx = \frac{2}{\theta^2}\int_0^\infty \left(x^{-3}e^{-(\theta x)^{-2}}\right)^\rho dx = (2\theta)^{\rho-1}\rho^{\left(\frac{\rho+1}{2}\right)}\Gamma\left(\frac{\rho+1}{2}\right) \tag{23}$$

$$\int_0^\infty {I_2}^\rho dx = \int_0^\infty \left[\lambda_1 + 2(\lambda_2-\lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right]^\rho dx$$

Finally, we have,

$$E_R(\rho) = \frac{1}{1-\rho} log\left[Z^{**}\int_0^\infty \left[\lambda_1 + 2(\lambda_2-\lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right]^\rho dx\right]$$

$$Z^{**} = (2\theta)^{\rho-1}\rho^{\left(\frac{\rho+1}{2}\right)}\Gamma\left(\frac{\rho+1}{2}\right)$$

3.3 Order Statistics

Let $X_1, X_2, . . . , X_n$ be a random sample of size n from CTF distribution and let $Z_{j:n}$ denote the j^{th} order statistics, then the pdf of $x_{j:n}$ is given by

$$f_{j:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(z)[F(z)]^{j-1}[1-F(z)]^{n-j} \tag{24}$$

Applying binomial expansion in (24), we have

$$f_{j:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(z)\sum_{k=0}^{\infty}\binom{n-j}{k}[F(z)]^{j+k-1} \tag{25}$$

Using equation (7) and (8) in (25), we obtain the j^{th} order statistics of CTF distribution given by

$$f_{j:n}(x) = \frac{n!\,\gamma Q_x}{(i-1)!\,(n-i)!}\frac{2}{\theta^2 x^3}e^{-(\theta x)^{-2}}\left[\lambda_1 + 2(\lambda_2-\lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right]$$

where,

$$Q_x = \sum_{k=0}^{\infty}\binom{n-j}{k}\left[\lambda_1 e^{-(\theta x)^{-2}} + (\lambda_2-\lambda_1)\left(e^{-(\theta x)^{-2}}\right)^2 + (1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^3\right]^{j+k-1}$$

We can obtain the first orders statistics by taking $j = 1$, also we can obtain the n^{th} order by taking $j = n$ in equation (25)

4.0 Maximum likelihood estimation for parameters of CTF distribution

Let $X_1, X_2, . . ., X_n$ be a random sample taken from CTF with parameters $\omega(\zeta, \gamma, \lambda_1, \lambda_2)$. The likelihood function is given by

$$L(\mathcal{Z}\,|\,\underline{x}) = \prod_{i=1}^{n}\frac{2}{\theta^2 x^3}e^{-(\theta x)^{-2}}\left[\lambda_1 + 2(\lambda_2-\lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right] \tag{26}$$

And the log-likelihood function is given by

$$l(\mathcal{Z}\,|\,\underline{x}) = -2nlog(\theta) + \theta^{-2}\sum_{i=1}^{n} x^2 \sum_{i=1}^{n}\left[\lambda_1 + 2(\lambda_2 - \lambda_1)e^{-(\theta x)^{-2}} + 3(1-\lambda_2)\left(e^{-(\theta x)^{-2}}\right)^2\right]$$

To obtain a numerical solution for the values of the estimates of CTF distribution we may employ software such as R, Maple, OX Program etc.

4.1 Application to lifetime data

Here, we show the flexibility of Cubic Transmuted Inverse Rayleigh distribution using a breast cancer data. The data represent 121 breast cancer patients' survival times during a specific period from 1929 to 1938. The data source Ramos et al. (2013) and Tahir et al. (2014) studied these datasets. The observations are listed as follows: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4,10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7,16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9,21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1,30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0,38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0,154.0. Then we fit of the Cubic Transmuted Inverse Rayleigh model is compared with the inverse Rayleigh model which is the baseline model. The maximum likelihood estimation procedure was used to determine the estimates of the parameters of the models considered. We compute statistical measures which represent the measures of goodness of fits and this includes Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hana Quin Information Criterion (HQIC). The model with the smallest AIC, BIC, CAIC, and HQIC is assumed to be the best fit model in the class of models considered. Table 1 shows that the breast cancer data is over-dispersed, positively skewed. Figure 4 shows that the Breast cancer data contains no outlier observation and it has a wider spread.

Table 1 Exploratory data analysis of Breast cancer data

$Min.$	q_1	$median$	$mean$	q_3	max	$Range$	$Skewness$	$Kurtosis$	$variance$
0.0004	0.0095	0.7405	0.9765	1.4122	3.8474	3.8470	1.0432	3.4021	1244.46

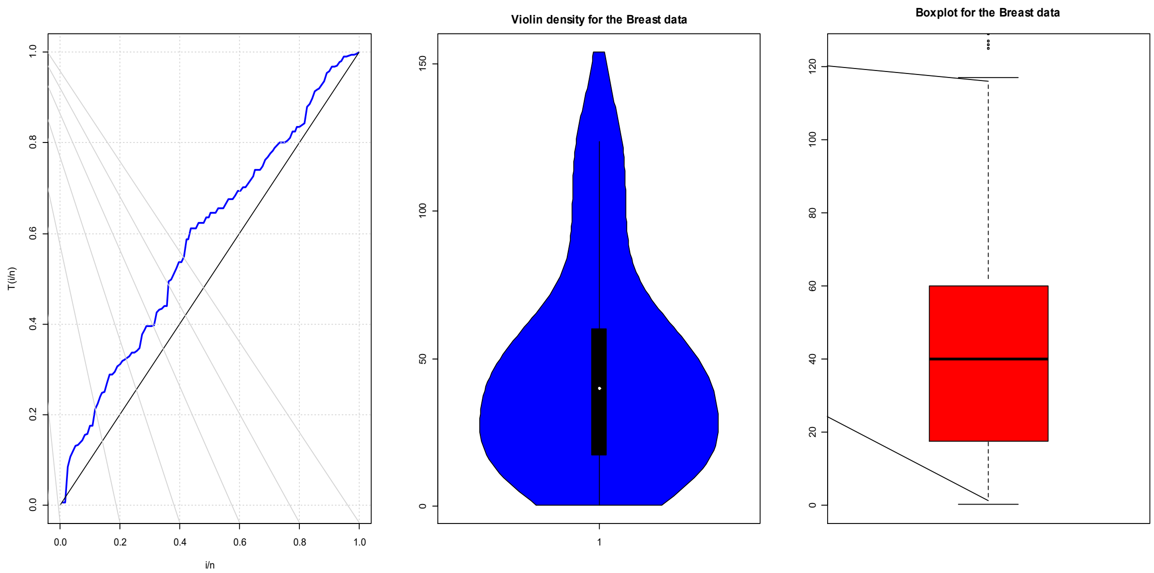


Figure 4: TTT plot, Violin plot and Kernel density plot for the breast cancer data

Table 2. MLEs and the measures of goodness of fit

$Model$	$Parameters$	$-l$	AIC	BIC	$CAIC$	$HQIC$
$CTIR$	$\lambda_1 = 1.2751(0.9015)$ $\lambda_2 = 1.3774(10.8802)$ $\theta = 5.1346(0.4401)$	534.80	1075.59	1083.98	1075.80	1079.0
IR	$\theta = 5.3385(0.4853)$	1092.55	2187.10	2189.89	2187.13	2188.23

The ML estimates along with their standard error (SE) of the model parameters are provided in the same tables, the analytical measures including; minus log-likelihood $(-log\ L)$, Akaike information Criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC). The fit of the proposed CTF distribution is compared its sub-model.

5.0 Conclusion

It is clear in this study that the new model developed show the potentiality of a process or phenomenon under consideration. The new distribution was developed using the cubic transmuted family of distributions introduced in the literature. Some mathematical properties of the new developed generalized distribution were studied, and also the parameters of the developed model are estimated using the maximum likelihood estimation procedure. An application of the newly developed model to breast cancer data demonstrates its attractiveness in modeling lifetime data.

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